A decomposition approach for facility location and relocation problem with uncertain number of future facilities

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Abstract

In this paper, we discuss two challenges of long term facility location problem that occur simultaneously; future demand change and uncertain number of future facilities. We introduce a mathematical model that minimizes the initial and expected future weighted travel distance of customers. Our model allows relocation for the future instances by closing some of the facilities that were located initially and opening new ones, without exceeding a given budget. We present an integer programming formulation of the problem and develop a decomposition algorithm that can produce near optimal solutions in a fast manner. We compare the performance of our mathematical model against another method adapted from the literature and perform sensitivity analysis. We present numerical results that compare the performance of the proposed decomposition algorithm against the exact algorithm for the problem.

Keywords: Facility location, uncertainty, p-median

1. Introduction

Facility location problems have been widely studied by many researchers on a variety of sectors. Examples can include public facilities such as schools and public libraries that are located to best serve the communities. Most of

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the time, community can be viewed as a group of people, where the initial demand for such facilities are known (Min, 1988). Locations of these facilities are not intended for a short term. These facilities should be able to serve the communities for a longer time, where we can expect changes in the demand of the communities.

As an example, we consider the population of 256 counties in the state of Texas. When we looked at the U.S. Census Data and compared the population in those counties for years 1990 and 2000, we observed that 22 counties had a population change - both increase and decrease - of more than 40%. Also, 73 counties had more than 20% population change during these years. Similar changes were observed all around U.S. (Cen).

Let us assume that we initially located facilities to serve those counties based on the demand in 1990. When we came to the year 2000, we would have observed that population change over a decade made some of the existing facilities being closer to lower demand, and some high demand areas being away from the existing facilities. Therefore, existing facilities may no longer be able to provide adequate service, which yields to an intolerable increase in total weighted distance traveled by the customers. In such situations, closing some of the existing facilities and opening new ones are essential and inevitable. For such problems, we need to determine the initial locations and a possible future relocation plan for the facilities in order to minimize the total traveling distance of the customers.

Locating facilities initially and relocating them in the future is a long term decision. In such long term decisions the number of future facilities usually may not be known for sure at the time of locating initial facilities. This uncertainty may happen due to various reasons. For example, suppose that the available budget or company policies limit the number of initial facilities to open at the beginning. However, additional budget or policy changes may allow the company to plan for a different number of facilities in the future. Also, the number of facilities in the future may depend on the success of the initial ones (Berman and Drezner, 2008). Therefore, the initial locations of the facilities should be determined considering the probability of change in the number of facilities in the future as well.

In this paper, we discuss two challenges of long term facility location problem that occur simultaneously; future demand change and uncertain number of future facilities. We introduce an integer programming (IP) model that minimizes the initial and expected future weighted distance of customers. Our model allows relocation for the future instances by closing some of the existing facilities and opening new ones, without exceeding a given budget. The rest of this paper is organized as follows. In Section 2, we present the relevant literature review. Section 3 describes the methodology, where we develop the mathematical model. In Section 4, we develop a decomposition algorithm as the proposed solution algorithm. In Section 5, numerical results are presented to show that our decomposition algorithm works well in both solution quality and time. We conclude the paper in Section 6.

2. Literature Review

P-median problem is a well known facility location problem and first introduced by Weber in the 1900's (Reese, 2006). The form that we are discussing today was developed by Hakimi in 1960's (Hakimi, 1964, 1965). Since then, solution techniques for the problem have become an abundant area of research. Various techniques such as relaxations, heuristics, and metaheuristics have been developed to solve the p-median problem (Reese, 2006). A recent paper introduced new approaches to improve the solution quality and CPU time of some of the existing techniques (Lim et al., 2009). The stochastic extensions of the p-median problem were also introduced in literature (Berman and Krass, 2002). The problem of locating p facilities currently and then locating new facilities in the future is discussed by Current et al. (1998). Because the number of future facilities is unknown, they used minimax regret criterion and compared it with the expected opportunity loss criterion. However, their objective function considers only the future scenarios, and assumes that the initially located facilities are not allowed to be closed in future scenarios. The conditional p-median problem solves the issue of locating a certain number of new facilities, with the knowledge of locations of the existing facilities (Drezner, 1995).

The p-median problem under uncertainty (Berman and Drezner, 2008) aims to locate p facilities initially, knowing that up to q additional facilities will be located in the future. Given the probabilities of locating $r \in \{0, 1, 2, \ldots, q\}$ more facilities, they formulated the problem on a graph, constructed the integer programming formulation, suggested heuristic solution techniques and extensively analyzed the case where $q = 1$. An important shortfall of this approach is that it does not allow facility closings. Both initial and future locations are determined considering a constant demand. However, as we discussed in the introduction section, due to demand changes over time, it may be necessary to close some of the existing facilities. The demand changes causing such relocations can easily be captured by forecasts and any uncertainty in the demand can be managed by improved forecasts (Owen and Daskin, 1998). From a computational point of view, their mathematical model is hard to solve especially for large scale problems and the proposed heuristic approaches are not suitable for cases with $q > 1$.

As we have mentioned, most of the papers in the literature concern opening new facilities only, whereas only a small fraction of these deals with closing facilities (Leorch et al., 1996). Wang et al. (2003) claims that it is not always realistic to consider closing existing facilities and opening new ones apart from each other. In the same manner, some authors considered relocation of facilities in dynamic environments such as the dynamic location allocation problem with facility relocation by Wesolowsky and Trusctott (1975). Their goal was to minimize the overall relocation and allocation costs considering the opening and closing costs of facilities. A relocation problem for public facilities was introduced with the solution of a real life problem in Min (1988). A fuzzy multi-objective model with constraints on budget and the maximum number of relocations per period was constructed to solve the problem. Supply chain point of view of the problem was also studied in the literature (Melo and Saldanha da Gama, 2005). They introduced a mixed integer programming (MIP) model to minimize the cost for a multi-commodity, multi-echelon, dynamic network by means of relocation of facilities and capacity transfers which all are performed in a given budget. Even though extra sources for the budget can be considered in real life, a fixed budget value has often been used in the literature. In our paper, assuming a fixed budget to control relocations, we consider different levels of available budget and perform sensitivity analysis so that the impact of different budget values on decision making could be observed.

Wang et al. (2003) aimed to minimize the total weighted distance between the facilities and customers, by closing some of the existing facilities and opening new ones, where they incur opening and closing costs. The main reason behind this relocation is customer demand change over time. One issue, which was not considered in their work, is the uncertainty of the number of facilities in the future. They assumed that the number of future facilities is given. However, the number of future facilities may not be certain for many reasons as we mentioned in the introduction section. Two approaches in the existing literature can be adapted for the cases where we have future demand change and uncertain number of future facilities.

In the first approach, the initial facility locations are optimized based on the current demand. When the demand change occurs in the future, the technique in (Wang et al., 2003) can be utilized to determine the relocations within a given budget in order to minimize the total weighted distance. Since this method does not consider the uncertainty of total number of facilities in the future, some facilities that are initially located might not be optimal for different future scenarios. This may lead to higher costs due to higher traveling distance, lower customer satisfaction and higher budget consumptions. The second approach is to find the best locations for initial facilities considering the possibility of adding more facilities (Berman and Drezner, 2008). However, this method does not consider closing any of the existing facilities and is unable to handle the future demand change. Therefore, this approach is not capable of dealing with the systems that are inherent to such a change. It is clear that, these challenges of relocation and uncertain number of future facilities can be observed in many sectors such as service industry, military bases, supply chain, etc. (Dell, 1998; Melo and Saldanha da Gama, 2005; Sathe and Miller-Hooks, 2005).

In order to overcome the limitations of the existing literature, we propose a solution approach that can determine the initial locations and future relocations of facilities where demand is subject to change and the number of future facilities is uncertain. Our aim is to minimize the initial and expected future weighted distances without exceeding the given budget for opening and closing facilities.

3. Methodology

3.1. Problem Definition

Suppose that we have a connected, undirected network, $\mathbb{N} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V} = \{v_1, \ldots, v_n\}$ is the set of n vertices and $\mathbb{E}$ is the set of edges. Let $w_i \geq 0$ be the initial demand at vertex v_i, and $\lambda_i \geq 0$ be the demand forecast in the future at vertex v_i. The shortest path between v_i and v_j is denoted as d_{ij}. Parameter p is the initial number of facilities and q is the upper limit for the increase in number of future facilities. The probability that $r \in [0, q]$ facilities will be added is estimated as α_r such that $\sum\limits_{r=0}^{q} \alpha_r = 1$. Due to the changes in the future demand in the network, we are allowed to perform relocations; we can close some of the existing facilities and open new facilities. Opening cost for a facility at v_j is denoted as o_j and closing cost is denoted by c_j. Parameter b is the total available budget for opening and closing facilities.

We will discuss two approaches to handle the facility location and relocation problem presented above. The first approach is adapted from (Wang et al., 2003) and because of its deterministic nature we label it Facility Location and Relocation Problem - Deterministic (FLRP-D). Our proposed approach solves the problem under uncertain number of future facilities and is called Facility Location and Relocation Problem - Uncertainty (FLRP-U).

3.1.1. FLRP-D

The notation $\mathbb{V}_1$ is the set of existing facilities and $\mathbb{V}_2$ is the set of potential facility sites, where $\mathbb{V} = \mathbb{V}_1 \cup \mathbb{V}_2$. In this method, $(p + r)$ will be the total number of desired facilities where the problem will be solved for all $r \in [0, q]$. We assume that initial facilities are located to the best possible sites based on the initial demand. After the demand change occurs, relocations can be performed using the following formulation. FLRP-D can be formulated as a Binary Integer Program (BIP) with two sets of decision variables which are:

$$\nu_j = \begin{cases} 1, & \text{if facility at } v_j \text{ is open, } \forall j \in \mathbb{V}, \\ 0, & \text{otherwise.} \end{cases}$$
$$x_{ij} = \begin{cases} 1, & \text{if demand at } v_i \text{ is assigned to facility at } v_j, \ \forall i, j \in \mathbb{V}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the problem formulation for FLRP-D for a given scenario r is

$$\min Z = \sum_{i \in \mathbb{V}} \lambda_i \sum_{i \in \mathbb{V}} d_{ij} x_{ij} \tag{3.1}$$
$$\text{s.t.} \sum_{j \in \mathbb{V}_1} c_j (1 - \nu_j) + \sum_{j \in \mathbb{V}_2} o_j \nu_j \leq b, \tag{3.2}$$
$$\sum_{j \in \mathbb{V}} \nu_j = p + r, \tag{3.3}$$
$$\sum_{j \in \mathbb{V}} x_{ij} = 1, \qquad \forall i \in \mathbb{V}, \tag{3.4}$$
$$x_{ij} \leq \nu_j, \qquad \forall i, j \in \mathbb{V}, \tag{3.5}$$
$$\nu_j, x_{ij} \in \{0, 1\}, \quad \forall i, j \in \mathbb{V}. \tag{3.6}$$

The objective of the formulation is to minimize the total weighted distance. Constraint (3.2) is to limit that opening and closure of the facilities are performed within the given budget. Constraint (3.3) makes sure that exactly $p + r$ facilities are located. Constraint (3.4) states that demand at each v_i is assigned to a facility. Constraint (3.5) makes sure that we cannot assign the demand at v_i to v_j unless a facility is located at v_j.

3.1.2. FLRP-U

In FLRP-U, our aim is to find the best locations for the initial facilities, and identify the possible relocations in the future knowing that number of future facilities may increase by $r, r \in [0, q]$. The relocations are maintained by closing some of the facilities that were located initially and opening new ones and the objective is minimizing the total of the initial weighted distance and the expected future weighted distance traveled by customers. Unlike FLRP-D, location and relocation of facilities are optimized considering the future demand change and uncertain number of future facilities, simultaneously. FLRP-U can also be formulated as BIP. For FLRP-U formulation, there are six sets of decision variables and they are defined as follows:
The number of additional facilities is given as r, for $r = 0, \ldots, q$.

$$\xi_j = \begin{cases} 1, & \text{if one of the initial } p \text{ facilities is located at } v_j \text{, for } j = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\pi_{ij} = \begin{cases} 1, & \text{if demand at } v_i \text{ is assigned to facility at } v_j \text{, for } i, j = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

$$y_j^r = \begin{cases} 1, & \text{if facility at } v_j \text{ is selected to open, for } j = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

$$z_j^r = \begin{cases} 1, & \text{if facility at } v_j \text{ is selected to close, for } j = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

$$s_j^r = \begin{cases} 1, & \text{if facility at } v_j \text{ is open (facility exists), for } j = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

$$x_{ij}^r = \begin{cases} 1, & \text{if demand at } v_i \text{ is assigned to facility at } v_j,\ i, j = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Note that defining the binary allocation variables (π_{ij} for FLRP-D and x_{ij}^r for FLRP-U) as continuous variables may result in a binary solution in some problem instances. However, this is not true for all cases as reported in Rosing et al. (1979).

Then, the problem formulation is

$$\min Z = \sum_{r=0}^{q} \alpha_r \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} d_{ij} x_{ij}^r + \sum_{i=1}^{n} w_i \sum_{j=1}^{n} d_{ij} \pi_{ij} \tag{3.7}$$

$$\text{s.t.} \sum_{j=1}^{n} \left(c_j z_j^r + o_j y_j^r \right) \leq b, \qquad r = 0, \ldots, q, \tag{3.8}$$

$$\sum_{j=1}^{n} \xi_j = p, \tag{3.9}$$

$$\sum_{j=1}^{n} s_j^r = p + r, \quad r = 0, \ldots, q, \tag{3.10}$$

$$\sum_{j=1}^{n} \pi_{ij} = 1, \qquad i = 1, \ldots, n, \tag{3.11}$$

$$\sum_{j=1}^{n} x_{ij}^r = 1, \qquad i = 1, \ldots, n, \; r = 0, \ldots, q, \tag{3.12}$$

$$\pi_{ij} \leq \xi_j, \qquad i, j = 1, \ldots, n, \tag{3.13}$$

$$x_{ij}^r \leq s_j^r, \qquad i, j = 1, \ldots, n, \; r = 0, \ldots, q, \tag{3.14}$$

$$s_j^r - \xi_j - y_j^r + z_j^r = 0, \qquad j = 1, \ldots, n, \; r = 0, \ldots, q, \tag{3.15}$$

$$y_j^r + z_j^r \leq 1, \qquad j = 1, \ldots, n, \; r = 0, \ldots, q, \tag{3.16}$$

$$\xi_j, \pi_{ij} \in \{0, 1\}, \; i, j = 1, \ldots, n, \tag{3.17}$$

$$y_j^r, z_j^r, s_j^r, x_{ij}^r \in \{0, 1\}, \; i, j = 1, \ldots, n, \; r = 0, \ldots, q. \tag{3.18}$$

The objective of the problem is to minimize the total weighted distance which is the summation of current weighted distance and expected future weighted distance. Constraint (3.8) is the budget limitation. Constraints (3.9) and (3.10) make sure that p facilities are located initially, and $p + r$ facilities are located in the future. Constraint (3.11) and (3.12) make sure that we assign the demand at each v_i to a facility in the initial case and future scenarios, respectively. Constraint (3.13) and (3.14) ensure that we cannot assign the demand at v_i to a facility unless a facility is located at v_j in the initial case and future scenarios, respectively. Constraint (3.15) sets the conditions for existence of a facility in future scenarios. Constraint (3.16) prevents simultaneous opening and closing of a facility in each scenario. If we omit this constraint, some optimal solutions may suggest that a facility be opened and closed at the same time. Mathematically, it should not matter if there is an ample budget to do so and it would not affect the objective function value. However, in reality, it is not reasonable to open and close the same facility at the same time. Therefore, having this constraint will ensure a more accurate and practical solution.

3.2. Problem Complexity

Theorem 1 below states that FLRP-D is *NP-Hard*. We now attempt to show complexity of FLRP-U by reducing the problem to FLRP-D or its equivalent formulation.

Theorem 1. *FLRP-D is NP-hard. (Wang et al., 2003).*

To find an equivalent formulation of FLRP-D, we first reformulate FLRP-D with different variables and parameters following a similar structure to the formulation of FLRP-U.
Let ξ_j be a parameter that assumes the value of 1 if one of the p facilities is located at v_j initially and 0 otherwise. Then the decision variables are as follows:

$$y_j = \begin{cases} 1 & \text{if facility at } v_j \text{ is selected to open in the future,} \\ & \text{for } j = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

$$z_j = \begin{cases} 1 & \text{if facility at } v_j \text{ is selected to close in the future,} \\ & \text{for } j = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

$$s_j = \begin{cases} 1 & \text{if facility at } v_j \text{ is open (facility exists) in the future,} \\ & \text{for } j = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if demand at } v_i \text{ is assigned to facility at } v_j \text{ in the future,} \\ & \text{for } j = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Then the reformulation for FLRP-D for a given scenario r is as follows:

$$\min Z = \sum_{i \in \mathbb{V}} \lambda_i \sum_{i \in \mathbb{V}} d_{ij} x_{ij} \tag{3.19}$$

$$\text{s.t.} \sum_{j=1}^{n} (c_j z_j + o_j y_j) \leq b, \tag{3.20}$$

$$\sum_{j=1}^{n} s_j = p + r, \quad r = 0, \ldots, q, \tag{3.21}$$

$$s_j - \xi_j - y_j + z_j = 0, \quad j = 1, \ldots, n, \tag{3.22}$$

$$y_j + z_j \leq 1, \quad j = 1, \ldots, n, \tag{3.23}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, n, \tag{3.24}$$

$$x_{ij} - s_j \leq 0, \quad i, j = 1, \ldots, n, \tag{3.25}$$

$$y_j, x_{ij} \in \{0, 1\}, \quad i, j = 1, \ldots, n. \tag{3.26}$$

Propositon 1. *The original formulation and the reformulation of FLRP-D are equivalent.*

Proof. The objective functions of the original formulation of FLRP-D and its reformulation are the same. Therefore, we will show the equivalence of the two formulations, by performing the following steps. If a facility is open at node j, variables s_j and ν_j are equal to 1 and 0, otherwise, in their corresponding formulations. Therefore, we can put s_j instead of ν_j in the original formulation. By using the constraint (3.22) in the reformulation, we can rewrite the constraint (3.2) in the original formulation. In the first part of the constraint, for $\mathbb{V}_1$, since the only possible operation in $\mathbb{V}_1$ is closing a facility, $s_j = \xi_j - z_j$. In the second part, for $\mathbb{V}_2$, since the only possible operation in $\mathbb{V}_2$ is opening a new facility, $s_j = \xi_j + y_j$. Then, the constraint in the original formulation can be modified as follows:

$$\sum_{j \in \mathbb{V}_1} c_j \left(1 - \xi_j + z_j\right) + \sum_{j \in \mathbb{V}_2} o_j (\xi_j + y_j) \leq b \tag{3.27}$$

As we know that $\xi_j = 1$ for $\mathbb{V}_1$ and $\xi_j = 0$ for $\mathbb{V}_2$, the constraint becomes

$$\sum_{j \in \mathbb{V}_1} c_j\left(z_j\right) + \sum_{j \in \mathbb{V}_2} o_j(y_j) \leq b \tag{3.28}$$

In order to combine the two parts, we add the constraint $y_j + x_j \leq 1$ to make sure that at most one of those actions (opening and closing) can be performed, which ensures that there is no intersection between $\mathbb{V}_1$ and $\mathbb{V}_2$. $\square$

Corollary 1. *FLRP-U is NP-hard.*

Proof. If we formulate FLRP-U for $r = 0$ and relax the constraints (3.9), (3.12) and (3.14), the problem reduces to the reformulation of FLRP-D. Due to the increase in number of variables and constraints, solving FLRP-U for multiple scenarios is much harder than solving the problem for a single scenario, when $r = 0$. Therefore, by the Theorem 1 and Proposition 1 we claim that FLRP-U is NP-hard. $\square$

4. Solution Algorithm

As we discussed, FLRP-U is NP-hard and the problem size grows rapidly when we increase the number of facilities as well as future scenarios. Therefore, BIP formulation of the problem becomes very difficult to solve optimally, especially for large instances. To be able to solve the problem in a timely manner, we analyzed the formulation and observed the block-angular structure of the problem, which is suitable for a decomposition approach (Sweeney and Murphy, 1979). Then, we developed a decomposition algorithm to solve FLRP-U, where initial and future scenarios for facility locations demonstrate the blocks and the rest of the constraints form the bridge constraints. The block-angular structure of the problem can be observed better in the following re-arranged formulation. The first three sets of constraints are bridge constraints and initial and future scenarios for facility locations are the blocks connected by the bridge constraints.

$$
\begin{array}{llllll}
\text{Min} & \sum_{i=1}^{n} w_i \sum_{j=1}^{n} \pi_{ij} d_{ij} & +\alpha_0 \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} x_{ij}^0 d_{ij} & + \ldots & +\alpha_q \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} x_{ij}^q d_{ij} & \\
\text{s.t.} & & & & & \\
& -\xi_j & +s_j^0 & & & -y_j^0 + z_j^0 \quad = 0 \\
& & & \ddots & & \vdots \\
& \xi_j & & & +s_j^q & -y_j^q + z_j^q \quad = 0 \\
& & & & & \sum_{j=1}^{n} o_j y_j^r + \sum_{j=1}^{n} c_j z_j^r \leq b \\
& & & & & y_j^r + z_j^r \quad \leq 1 \\
& \sum_{j=1}^{n} \xi_j & & & & = p \\
& \sum_{j=1}^{n} \pi_{ij} & & & & = 1 \\
& -\xi_j + \pi_{ij} & & & & \leq 0 \\
& & \sum_{j=1}^{n} s_j^0 & & & = p \\
& & \sum_{j=1}^{n} x_{ij}^0 & & & = 1 \\
& & -s_j^0 + x_{ij}^0 & & & \leq 0 \\
& & & \ddots & & \vdots \\
& & & & \sum_{j=1}^{n} s_j^q & = p + q \\
& & & & \sum_{j=1}^{n} x_{ij}^q & = 1 \\
& & & & -s_j^q + x_{ij}^q & \leq 0.
\end{array}
$$

For the ease of illustration, the following variable and parameter substitutions will be used for the decomposition algorithm. Let t_k and u be binary decision variables. Variable t_k is substituted for variables $\xi_j, \pi_{ij}, s_j^r, x_{ij}^r$ and k is the index for each block i.e. $t_1 = [\xi_1, \ldots, \xi_n, \pi_{11}, \ldots, \pi_{nn}]$, $t_2 = [s_1^0, \ldots, s_n^0, x_{11}^0, \ldots, x_{nn}^0]$, $\ldots$, $t_{q+2} = [s_1^q, \ldots, s_n^q, x_{11}^q, \ldots, x_{nn}^q]$. The variable u is substituted for variables y_j^r and z_j^r i.e. $u = [y_1^0, \ldots, y_n^q, z_1^0, \ldots, z_n^q]$. Let A and B represent the constraint coefficients and right hand side values of the constraints, respectively, and C represent the objective function coefficients.

Then FLRP-U can be re-written as follows:

$$
\begin{array}{lllllllll}
\text{Min} & C_1 t_1 & +C_2 t_2 & +\ldots & +C_{q+2} t_{q+2} & & & \\
\text{s.t.} & & & & & & & \\
& A_{011} t_1 & +A_{012} t_2 & +\ldots & +A_{01(q+2)} t_{q+2} & +A_{01(q+3)} u & = & B_{01} \\
& & & & & A_{02(q+3)} u & \leq & B_{02}, \\
& & & & & A_{03(q+3)} u & \leq & B_{03}, \\
& A_{11} t_1 & & & & & = & B_{11}, \\
& A_{12} t_1 & & & & & = & B_{12}, \\
& A_{13} t_1 & & & & & \leq & B_{13}, \\
& & A_{21} t_2 & & & & = & B_{21}, \\
& & A_{22} t_2 & & & & = & B_{22}, \\
& & A_{23} t_2 & & & & \leq & B_{23}, \\
& & & \ddots & & & \vdots & \\
& & & & A_{(q+2)1} t_{q+2} & & = & B_{(q+2)1}, \\
& & & & A_{(q+2)2} t_{q+2} & & = & B_{(q+2)2}, \\
& & & & A_{(q+2)3} t_{q+2} & & \leq & B_{(q+2)3}.
\end{array}
$$

If we apply Lagrangian relaxation to the bridge constraints (with multiplier μ), we will obtain (q+2) sub-problems as follows:

$$
(SP_k) \quad
\begin{array}{ll}
\text{Min} & (C_k - \mu A_{0k}) t_k \\
\text{s.t.} & \\
& A_{k1} t_k = B_{k1}, \\
& A_{k2} t_k = B_{k2}, \\
& A_{k3} t_k \leq B_{k3}, \\
& t_k \in \{0, 1\}.
\end{array}
$$

Let t^* denote the optimal solution for each subproblem, then a lower bound for the original problem can be obtained by the following equation (Sweeney and Murphy, 1979).

$$
z_l = (C_1 - \mu A_{01}) t_1^* + (C_2 - \mu A_{02}) t_2^* + \ldots + (C_{q+2} - \mu A_{0(q+2)}) t_{q+2}^* + \mu B_0 \quad (4.1)
$$

A good lower bound can be obtained by solving the LP relaxation of the sub-problems since LP relaxation of p-median problems often leads to optimal or very close to optimal solutions (Rosing et al., 1979).

Let s_k be the set of columns that are included in the master problem for subproblems $k = 1, \ldots, (q+2)$. Let $\tau_{s_k}^k$ be 1 if the corresponding column is in the optimal master problem solution and 0 otherwise. Then we can construct the master problem as follows:

$$
\begin{array}{llllll}
(MP) & & & & & \\
\text{Min} & \sum_{s_1}(C_1 t^1_{s_1})\tau^1_{s_1} & + \sum_{s_2}(C_2 t^2_{s_2})\tau^2_{s_2} & + \ldots + & \sum_{s_{q+2}}(C_{q+2} t^{q+2}_{s_{q+2}})\tau^{q+2}_{s_{q+2}} & \\
\text{s.t.} & & & & & \\
& \sum_{s_1} A_{011} t^1_{s_1} & + \sum_{s_2} A_{012} t^2_{s_2} & + \ldots + & \sum_{s_{q+2}} A_{01(q+2)} t^{q+2}_{s_{q+2}} & + A_{01(q+3)} u = B_{01}, \\
& & & & & A_{02(q+3)} u \leq B_{02}, \\
& & & & & A_{03(q+3)} u \leq B_{03}, \\
& \sum_{s_1} \tau^1_{s_1} & & & & = 1, \\
& & \sum_{s_2} \tau^2_{s_2} & & & = 1, \\
& & & \ddots & & \\
& & & & \sum_{s_{q+2}} \tau^{q+2}_{s_{q+2}} & = 1.
\end{array}
$$

The parameter B_{01}, which is the right-hand-side value of the bridge constraints that connect the subproblems, is equal to 0. Therefore, equation (4.1) implies that selecting the Lagrangian coefficient (μ) as a nonzero value may not have a big impact on the lower bound quality. So we select (μ) to be 0, and then each subproblem becomes a weighted p-median problem where the first subproblem has the initial weights of the vertices (w_i), and the rest of the subproblems have the multiplication of scenario probability by future weights of the vertices ($\alpha_k \lambda_i$) for $k = 2, \ldots, (q+2)$, $i = 1, \ldots, n$.
The column sets for the master problem are obtained by solving each subproblem using a modified version of Discrete Lloyd Algorithm (DLA), a heuristic solution technique that can generate good upper bounds for the p-median problem (Lim et al., 2009). DLA can be used for problems that are on a real network with uniform vertex weights. The algorithm starts with an initial set of medians, and divides the network into p clusters by assigning each vertex to its closest median. For each cluster, center of gravity is determined and projected to the closest vertex in the network, which will construct an updated set of medians. The procedure is repeated until there is no change in the median locations. Since our subproblems are weighted p-median problems, we make a slight modification to their algorithm when calculating the center of gravity by incorporating the vertex weights.
The structure of our proposed solution algorithm is explained in Figure 1.

Initialize: $i \leftarrow 0$,
$m \leftarrow$ Maximum number of iterations,
$\delta^* \leftarrow$ Desired dual gap.

1. Solve the LP relaxation of (SP_k) for $k = 1, \ldots, q+2$ and calculate the lower bound, z_l, (4.1).

do{

2. Create multiple feasible solutions for (SP_k) for $k = 1, \ldots, q+2$ using DLA.
   Add those feasible solutions as columns to (MP).
3. Solve (MP) and obtain the upper bound, z_u.
4. Calculate the actual dual gap,
   $\delta_i = \frac{z_u - z_l}{z_l}$.
5. $i \leftarrow i + 1$.

}while$\{(\delta_i > \delta^*) and (i \leq m)\}$

Figure 1: Decomposition algorithm for FLRP-U

The algorithm starts by initializing the parameters. The lower bound (z_l) for FLRP-U is calculated by adding up the objective function values of the LP relaxation of each subproblem. Then, multiple feasible solutions for each subproblem are created using DLA. An upper bound for FLRP-U (z_u) is obtained by solving the master problem and the actual dual gap is calculated by $(z_u - z_l)/z_l$. If the actual dual gap is greater than the desired dual gap and maximum number of iterations has not been reached, we create more solutions for the subproblems and continue the algorithm.

5. Numerical Results

In this section we present our numerical results for the methods discussed in Sections 3 and 4. We will first illustrate FLRP-U on a small network and present the numerical results that are used to compare FLRP-U and FLRP-D. Then, we will perform sensitivity analysis for FLRP-U. We will also show the results that compare the performance of our algorithm with the exact solution technique, the mathematical model we introduced in Section 3.1.2. All numerical results presented in this section were run on a Pentium 4 Xeon 3.6 Ghz machine with 4 GB RAM.

5.1. An Illustration with 1-median problem

A small example for FLRP is illustrated in Figure 2. The network has six nodes and nine arcs. Numbers near the nodes are the initial and forecasted future demand $(initial, future)$. Numbers on the arcs represent the distances between the nodes. Suppose we need to locate one facility initially, and

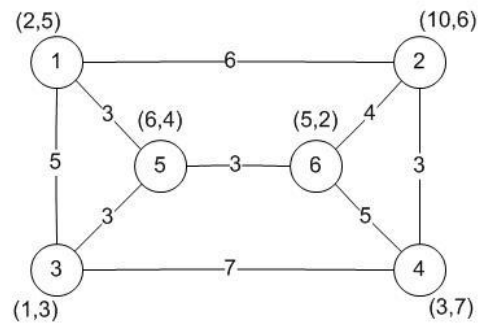


Figure 2: An illustrative 6 node network

we may locate up to two more facilities in the future, i.e. $q = 2$. The probability of adding one facility and two facilities are assumed to be 0.3, respectively, while the probability of adding no facilities is 0.4. A budget of 500 is available for the facility opening and closing operations. Based on the initial weights, optimum one median is located at node 6. Based on FLRP-D, the optimal decision when $r = 0$ is to close the facility at node 6 and open a new facility at node 2. The optimal solution for $r = 1$ is to keep the existing facility at node 6 and open another one at node 4. For $r = 2$, it is to keep the existing facility at node 6 and to open two new facilities at nodes 3 and 4. On the other hand, FLRP-U proposes to open the initial facility at node 2. In the future, when $r = 0$, the optimal decision is to keep that facility. When $r = 1$, FLRP-U suggests that we keep the existing facility and open a new facility at node 5. When $r = 2$, we should again keep the existing facility at node 2, and open new facilities at nodes $1, 3$. The total weighted distance comparison for FLRP-D and FLRP-U is shown in Table 1. Table 2 shows the expected budget consumption for both methods.

Expected weighted distance and expected budget consumption is calculated by multiplying the probability of each scenario by the corresponding weighted distance and budget consumption, respectively. As can be seen from these tables, FLRP-U yields a better objective function value and lower budget consumption. Since this is a small example with a small budget, FLRP-U did not propose closing of the existing facility in any scenario. However, for larger cases, it considers the relocation opportunity to find the best solutions.

Table 1: Total weighted distance for both methods

Method	Weighted Distance					
	initial	r=0	r=1	r=2	expected	total
FLRP-D	91	117	78	55	86.7	177.7
FLRP-U	93	117	51	41	74.4	167.4

Table 2: Budget consumption for both methods

Method	Budget Consumption			
	r=0	r=1	r=2	expected
FLRP-D	280	259	494	337.9
FLRP-U	0	293	497	237

5.2. FLRP-U vs. FLRP-D

As mentioned previously, for FLRP-D we assumed that initial facilities are located to the best possible sites, based on the initial weights. For comparison purposes, future weighted distances for each scenario are used to calculate the expected weighted distance. Summation of expected weighted distance and initial weighted distance yields the total weighted distance, which forms the main objective function.

The FLRP-U was tested on 20 randomly generated networks with 250 nodes in each network. Table 3 shows the parameters that are generated randomly from their corresponding uniform distributions. Table 4 shows different instances for q values and corresponding probability for each instance. Our experiments consist of five budget levels $500, 750, 1000, 1500$ and 3000. We also ran the experiments with the same parameters for FLRP-D for comparison purposes. All experiments are coded in GAMS (Brook et al., 2009)

Table 3: Parameters

Parameter	Uniform Distribution
Initial Demand (w)	[100, 200]
Future Demand (λ)	[50, 250]
Opening Cost (o)	[200, 300]
Closing Cost (c)	[50, 100]

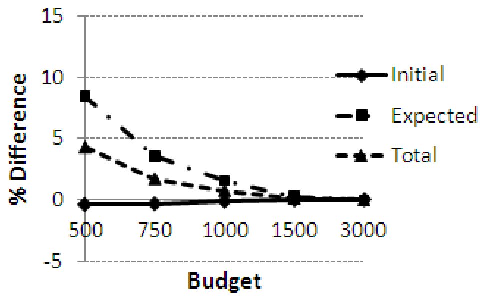
Table 4: Future instances and scenario probabilities

Upper Limit on Number of Future Facilities (q)	Scenarios		
	$r = 0$	$r = 1$	$r = 2$
0	1	-	-
1	0.5	0.5	-
2	0.4	0.3	0.3

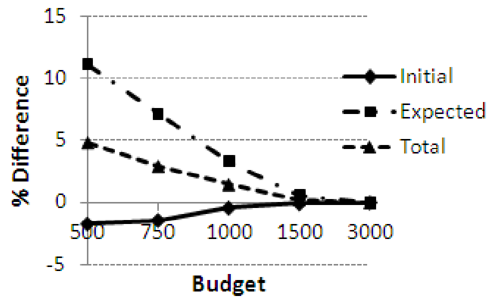
and solved by CPLEX. Percent difference of the objective function values are calculated by subtracting average objective function values of FLRP-U from FLRP-D and dividing by FLRP-U for all scenarios and budget values.

Figures 3(a), 3(b), and 3(c) illustrate initial weighted distance, expected weighted distance, and the total weighted distance differences between FLRP-U and FLRP-D when $q = 0$, $q = 1$, and $q = 2$, respectively. Initial weighted distance of FLRP-U is always higher than the initial weighted distance of FLRP-D for all cases. This is not surprising because initial weighted distance for FLRP-D is the optimal solution based on the initial weights. With respect to the initial weights; the difference between FLRP-D and FLRP-U becomes smaller for all scenarios, as we increase the budget. This gap also decreases when the upper bound for the number of future facilities decreases.

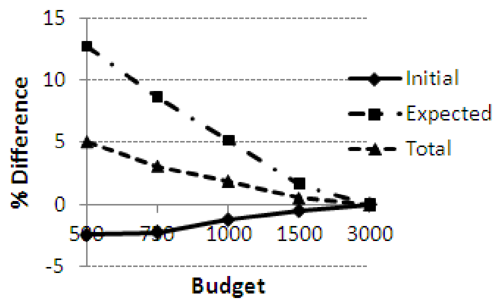
For the expected weighted distance, we again observe that FLRP-U produces better results in all instances. Related with the individual scenarios, it performs better when we have a lower budget. Opposite to the initial weighted distance values, the gap between the methods increases when the upper bound for the future facilities increases. We observe a larger difference between the results of FLRP-U and FLRP-D when $q = 2$, compared to $q = 1$ or $q = 0$. In FLRP-U, we let the initial weighted distance to increase slightly in order to handle the demand changes and uncertain number of future facilities. However, in FLRP-D, initial decisions are made without considering the future uncertainties, planning that necessary relocations would be performed



(a) q = 0



(b) q = 1



(c) q = 2

Figure 3: % Difference between FLRP-D and FLRP-U

after observing the demand change.
Figure 4 shows the difference of total weighted distance between FLRP-U and FLRP-D for different budget values. We make three observations based on these experiments. First, initial weighted distance for FLRP-D is always lower than or equal to FLRP-U. This gap decreases when either the budget is increased or the upper limit of the number of future facilities is decreased. Second, expected future weighted distance for FLRP-U is lower than or equal to FLRP-D. This gap increases when either we decrease the budget or increase the upper limit of the number of future facilities. Third, the two objectives explained above are contradicting. However, this contradiction results in favoring FLRP-U. Since the difference of the two methods for expected future weighted distance and initial weighted distance is higher in FLRP-D, FLRP-U yields better total weighted distance for the test problem instances. This gap also follows a similar pattern with the expected future weighted distance; increases when either the budget is decreased or the upper

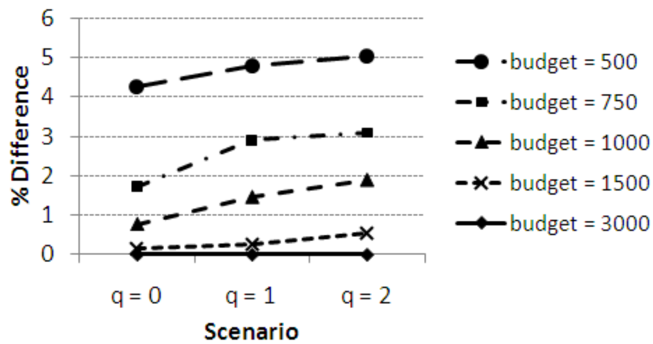


Figure 4: Total weighted distance difference between FLRP-D and FLRP-U wrt. future scenarios for different budget values

limit of number of future facilities is increased.

5.3. Sensitivity Analysis for FLRP-U

In this section we present sensitivity analysis for FLRP-U based on different budget values. This analysis will provide an insight about the trade-off between the allocated budget for relocations and total weighted distance traveled by customers to their closest facilities. As we discussed in Section 2, extra sources for the budget can be utilized in real life. Therefore, this analysis will also help a decision maker to explore various budget options considering the possibility of additional resources.

Sensitivity analysis was conducted on 20 randomly generated networks, each one having 250 nodes. The same parameters in Tables 3 and 4 were used. The budget for opening and closing facilities are set to be 500, 750, 1000, 1500, 3000 and 5000. Figure 5 shows a summary of the results. For $q = 0$, the line is almost flat, which means that allocating extra budget for such situations does not provide much improvement. This shows that, when there is no increase in the number of future facilities, the total of initial and expected future weighted distance does not depend too much on the available budget for relocations due to the fact that there is no uncertainty in the number of future facilities. For $q = 1$ and $q = 2$, the impact of extra budget is more evident, especially from 500 to 750. The rest of the improvement converges to a constant. This shows that, allocating extra budget for relocations beyond 3000 may not further improve the objective function value.

Overall, the increase in the budget from 500 to 750 provides significant decrease in the objective function value, especially for $q > 0$. Further increases

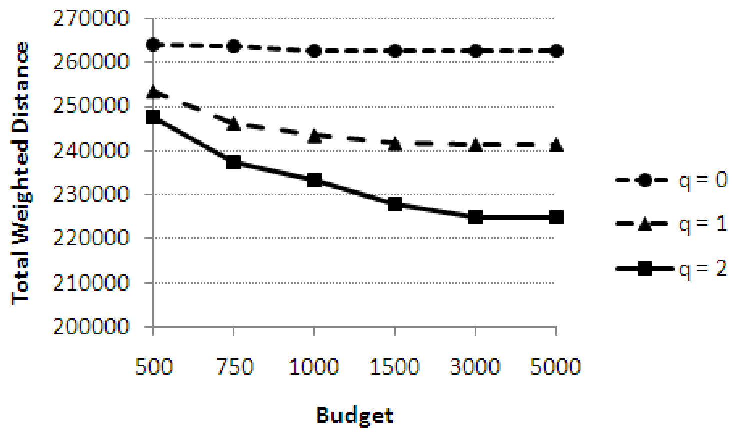


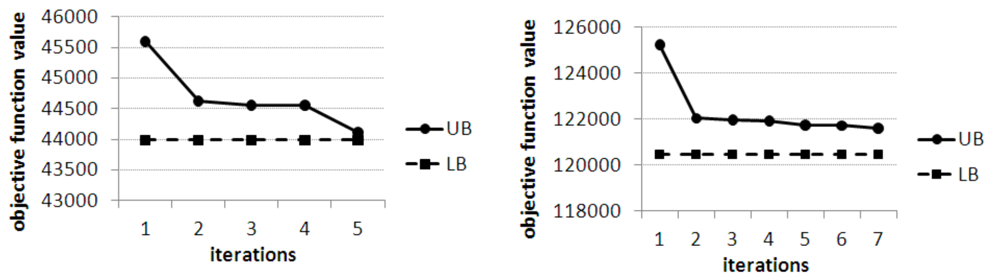
Figure 5: Trade-off curves between the budget and total weighted distance for q values

up to 3000 seem to be also beneficial, however the slope of the lines are getting smaller compared to the former. Overall, increasing budget seems to be most beneficial for larger value of q.

5.4. Decomposition Algorithm for FLRP-U

In order to test the computational performance of the IP Decomposition Algorithm, we generated 20 random networks of size 100, 250 and 500, respectively. The same parameters in Table 3 were used. The instance $q = 2$ in Table 4 is considered since it is more complicated than the other two instances. The budget for opening and closing facilities was set to 1200 for all instances. The number of initial columns created for the master problem and additional columns throughout the iterations are selected to be 10, 30, 50 for network sizes 100, 250 and 500, respectively. This column size selection was made in order to have feasible master problems. For each instance, the algorithm is run for desired dual gap levels 1%, 2%, 3%, 4%, 5% and 10%. We coded the algorithm in C++ and used CPLEX (IBM, 2009) to obtain the LB and solved the MP. Experiment instances are also solved by an exact solution method, which is the BIP formulation introduced in Section 3.1.2. The exact method is formulated in GAMS (Brook et al., 2009) and solved by CPLEX (IBM, 2009), and for each instance we set the relative termination tolerance to the same desired dual gap levels.

Figures 6(a) and 6(b) illustrate the convergence of the algorithm for two instances of network sizes 100 and 250. First, the lower bound is calculated



(a) n = 100 (b) n = 250

Figure 6: Convergence of the algorithm

using Equation 4.1 once and it is fixed throughout the iterations. Then the rest of algorithm attempts to minimize the upper bound by adding columns to the MP in each iteration.

Table 5 shows the average actual dual gap for all network sizes and desired dual gaps for both exact solution method and decomposition algorithm. The average solution time in CPU seconds and percent time gain for each case are also compared. The percent time gain is calculated by subtracting the average solution time of exact method from the decomposition algorithm and dividing by the average solution of decomposition algorithm. Standard deviation of the actual dual gap of both methods for each network size is plotted with respect to the desired dual gap level in Figure 7. This figure shows that variability in the actual dual gap for both methods is less in the decomposition algorithm for most of the instances.

As we can observe from Table 5, objective function values for both methods are within the desired dual gap which we used as a stopping criterion for both the exact method and decomposition algorithm. However, there is a substantial gain in its average CPU time of our decomposition algorithm compared to the exact approach, up to 86.5 %. From Table 5 and Figure 7, we can conclude that the average dual gap and standard deviation of both methods increased when we increase the desired dual gap level. We have also observed that the standard deviation for the decomposition algorithm is usually smaller than the exact solution method.

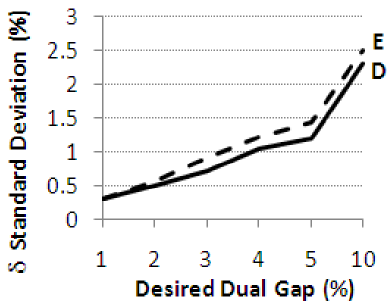
Table 5: Decomposition Algorithm and Exact Solution Method Comparison

	Stopping Criterion	Dual Gap (Average)		Solution Time (Average)		
		Exact	Decomp	Exact	Decomp	Gain (%)
n=100	1%	0.17%	0.38%	13.08	3.97	69.63
	2%	0.38%	1.20%	12.00	3.61	69.94
	3%	0.59%	1.78%	11.45	3.33	70.91
	4%	0.83%	2.30%	11.21	3.36	70.01
	5%	0.92%	2.36%	10.66	3.27	69.28
	10%	1.64%	2.76%	10.83	3.32	69.35
n=250	1%	0.18%	0.51%	468.04	117.53	74.89
	2%	0.30%	1.10%	443.59	96.73	78.19
	3%	0.56%	1.47%	414.83	95.62	76.95
	4%	0.76%	1.50%	402.38	94.84	76.43
	5%	0.81%	1.73%	402.08	95.16	76.33
	10%	1.46%	1.78%	373.74	95.66	74.41
n=500	1%	0.14%	0.76%	10028.00	1378.50	86.25
	2%	0.43%	1.11%	9049.90	1358.90	84.98
	3%	0.43%	1.72%	9102.60	1356.90	85.09
	4%	1.07%	1.64%	8583.20	1355.00	84.21
	5%	1.92%	2.01%	7756.40	1356.30	82.51
	10%	1.92%	2.32%	7503.54	1357.30	81.91

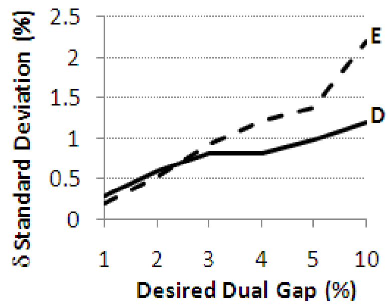
6. Conclusions

In this paper, we introduced the facility location problem that considers future demand changes as well as uncertain number of future facilities which we named FLRP-U. The objective is to minimize the sum of the current weighted distance and the expected future distance traveled by customers to their closest facilities without exceeding a given budget for opening and closing of facilities. As we discussed in Section 2, there are few approaches that consider closing of facilities, which is necessary to handle demand changes. Therefore, we presented a method that determines the initial locations and future relocations of facilities based on the fact that number of future facilities is not known exactly.

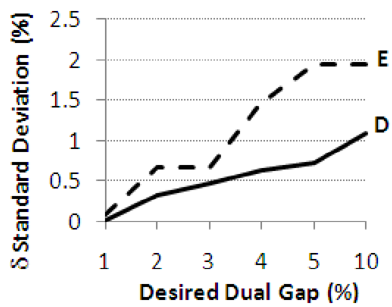
We presented an integer programming formulation of the problem and some numerical results that compare the performance of our method against an adapted version of another method found in literature, which was named



(a) n = 100



(b) n = 250



(c) n = 500

Figure 7: Standard Deviation of δ for Exact Solution Method (E) and Decomposition Algorithm (D) for network sizes $n = 100$, 250 and 500.

FLRP-D. Based on the average total weighted distance, FLRP-U outperformed FLRP-D in all scenarios, which shows that FLRP-U methodology is a valuable contribution to solve such problems.

We conducted sensitivity analysis that shows the impact of budget increase on total weighted distance. Those analysis can be utilized for making decisions about whether finding extra sources to increase the available budget for relocations is worthwile or not.

We introduced a decomposition algorithm for FLRP-U to ease the time to solve the problem for large scale instances and high uncertainty. We then presented numerical results that compare the objective function value and solution time of our decomposition algorithm with an exact solution method, concluding that our proposed method yields a significant time gain while satisfying the desired dual gap level.

Acknowledgment: This research has been supported by the Department of Homeland Security grant # 2008-ST-061-ND0002.

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